VECTOR CALCULUS

A project report submitted in partial fulfilment for the

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Submitted by

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CERTIFICATE

This is to certified that the project entitled **"VECTOR CALCULUS"** submitted by **ANNEPU DILEEP** for the award of Bachelor of Science (B.S.C.), Andhra University, Visakhapatnam, during the year 2021-22 is genuine record of the work done by him my supervision.

Place: Visakhapatnam

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DECLARATION

I, hereby declare that project entitled **"VECTOR CALCULUS"** is an original work done by me and submitted to the Department of mathematics and Statistics, Mrs. AVN College, Visakhapatnam, for the fulfillment of the 6^{th} Semester end examination. I also declare, that this or part of it has not been submitted to any other college for the award of degree.

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17. Vector Calculus with Applications

17.1 INTRODUCTION

In vector calculus, we deal with two types of functions: Scalar Functions (or Scalar Field) and Vector Functions (or Vector Field).

Scalar Point Function

A scalar function $F(x,y,z)$defined over some region R of space is a function which associates, to each point $P(x,y,z)$in R, a scalar value $F(P) = F(x,y,z)$. And the set of all scalars $F(P)$for all values of P in R is called the scalar field over R.

Precisely, we can say that scalar function defines a scalar field in a region or on a space or a curve. Examples are the temperature field in a body, pressure field in the air in earth's atmosphere.

Moreover, if the position vector of the point P is $\vec{r}$, then we may also write the scalar field as $F(P) = F(\vec{r})$. This notation emphasizes the fact that the scalar value $F(\vec{r})$ is associated with the position vector $\vec{r}$ in the region R.

E.g. 1) The distance $F(P)$ of any point $P(x,y,z)$ from a fixed point $P'(x',y',z')$ in the space is a scalar function whose domain of definition is the whole space and is given by $F(P) = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Also $F(P)$ defines a scalar field in space.

E.g. 2) The function $F(x,y,z) = xy^2 + yz + x^2$ for the point (x,y,z) inside the unit sphere $x^2 + y^2 + z^2 = 1$ is a scalar function and also is defines a scalar field throughout the sphere.

Note: In the physical problems, the scalar function F depends on time variable t in addition to the point P and then we write it as $F(P,t) = F(\vec{r},t) = F(x,y,z,t)$. The example of such a time dependent scalar function is the temperature distribution throughout a block of metal heated in such a way that its temperature varies with time.

Vector Point Function

A vector functions $\vec{F}(x,y,z)$ defined over some region R of space is a function which associates, to each point $P(x,y,z)$ in R, a vector value $\vec{F}(P) = \vec{F}(x,y,z)$ and the set of all vectors $\vec{F}(P)$for all points P in R is called the vector field over R

Moreover, if the position vector of the point P is $\vec{r}$, then we may write the vector field as $\vec{F}(P) = \vec{F}(\vec{r})$. This notation emphasizes the fact that the vector value $\vec{F}(\vec{r})$ is associated with the position vector $\vec{r}$ in the region R. Also the general form (component form) of the vector function is $\vec{F}(\vec{r}) = F_1(\vec{r})\hat{\imath} + F_2(\vec{r})\hat{\jmath} + F_3(\vec{r})\hat{k}$, where the components $F_1(\vec{r})$, $F_2(\vec{r})$ and $F_3(\vec{r})$ are the scalar functions.

E.g. 1) The function $\vec{F}(x,y,z) = 2xy\hat{\imath} + \sin x\,\hat{\jmath} + 3z^2\hat{k}$ for point $P(\ x,\ y,\ z\ )$ inside an ellipsoid $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1$ is a vector function and defines a vector field throughout the ellipsoid.

E.g. 2) The force field given by $\vec{F}(x,y,z) = x\hat{\imath} + 2y\hat{\jmath} + z^2\hat{k}$ is a vector field.

Note: Like the time dependent scalar field, time dependent vector field also exists. Such a field depends on time variable t in addition to the point in the region R and may be expressed as $\vec{F}(\vec{r},t) = F_1(\vec{r},t)\hat{\imath} + F_2(\vec{r},t)\hat{\jmath} + F_3(\vec{r},t)\hat{k}$, where F_1, F_2 and F_3 are scalar functions. An example of time dependent vector field is the fluid velocity vector in the unsteady flow of water around a bridge support column, because this velocity depends on the position vector $\vec{r}$ in the water and the time variable t and is given as $\vec{V}(\vec{r},t)$.

Vector Function of Single Variable

A vector function $\vec{F}$ of single variable t is a function which assigns a vector value $\vec{F}(t)$ to each scalar value t in interval $a \le t \le b$. In the component form, it may be written as $\vec{F}(t) = F_1(t)\hat{\imath} + F_2(t)\hat{\jmath} + F_3(t)\hat{k}$ where F_1, F_2 and F_3 are called components and are scalar functions of the same single variable t.

For example, the functions given by $\vec{F}(t) = t\,\hat{\imath} + \sin(t-2)\,\hat{\jmath} + \cos 3t\,\hat{k}$ and $\vec{G}(t) = t^2\hat{\imath} + e^t\hat{\jmath} + \log t\,\hat{k}$ are vector functions of a single variable t.

Limit of a Vector Function of Single Variable

A vector function $\vec{F}(t) = F_1(t)\hat{\imath} + F_2(t)\hat{\jmath} + F_3(t)\hat{k}$ of single variable t is said to have a limit $\vec{L} = L_1\hat{\imath} + L_2\hat{\jmath} + L_3\hat{k}$ as $t \to t_0$, if $\vec{F}(t)$ is defined in the neighborhood of t_0 and $\mathrm{Lt}_{t\to t_0}|\vec{F}(t) - \vec{L}| = 0$ or $\mathrm{Lt}_{t\to t_0}|F_1(t) - L_1| = \mathrm{Lt}_{t\to t_0}|F_2(t) - L_2| = \mathrm{Lt}_{t\to t_0}|F_3(t) - L_3| = 0$, then we write it as $\mathrm{Lt}_{t\to t_0}\vec{F}(t) = \vec{L}$.

Continuity of a Vector Function of Single Variable

A vector function $\vec{F}(t) = F_1(t)\hat{\imath} + F_2(t)\hat{\jmath} + F_3(t)\hat{k}$ of a single variable t is said to be continuous at $t = t_0$, if it is defined in some neighborhood of t_0 and $\mathrm{Lt}_{t\to t_0}\vec{F}(t) = \vec{F}(t_0)$.

Moreover, $\vec{F}(t)$ is said to be continuous at $t = t_0$ if and only if its three components F_1, F_2 and F_3 are continuous as $t = t_0$.

DIFFERENTIAL VECTOR CALCULUS

17.2 DIFFERENTIATION OF VECTORES

Differentiability of a Vector Function of Single Variable

A vector function $\vec{F}(t) = F_1(t)\hat{\imath} + F_2(t)\hat{\jmath} + F_3(t)\hat{k}$ of a single variable t defined over the interval $a \le t \le b$ is said to be differentiable at $t = t_0$ if the following limit exists.

$$\mathrm{Lt}_{t\to t_0}\frac{\vec{F}(t)-\vec{F}(t_0)}{t-t_0} = \vec{F}'(t_0)$$

And $\vec{F}'(t_0)$ is called the derivative of $\vec{F}(t)$ at $t = t_0$.

Also $\vec{F}(t)$ is said to be differentiable over the interval $a \le t \le b$, if it is differentiable at each of the points of the interval. In component form, $\vec{F}(t)$ is said to be differentiable at $t = t_0$ if and only if its three components are differentiable at $t = t_0$. In general, the derivative of $\vec{F}(t)$ is given by

$\vec{F}'(t) = \quad Lt_{t\to t_0}\frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t}$, provided the limit exists and in terms of components

$\vec{F}'(t) = F_1'(t)\hat{\imath} + F_2'(t)\hat{\jmath} + F_3'(t)\hat{k}$ or $\frac{d\vec{F}}{dt} = \frac{dF_1}{dt}\hat{\imath} + \frac{dF_2}{dt}\hat{\jmath} + \frac{dF_3}{dt}\hat{k}$.

In the similar manner, $\frac{d^2\vec{F}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{F}}{dt}\right), \frac{d^3\vec{F}}{dt^3} = \frac{d}{dt}\left(\frac{d^2\vec{F}}{dt^2}\right) = \frac{d^2}{dt^2}\left(\frac{d\vec{F}}{dt}\right)$.

Rules for Differentiation of Vector Functions

If $\vec{F}(t), \vec{G}(t)$ & $\vec{H}(t)$ are the vector functions and $f(t)$ is a scalar function of single variable t defined over the interval $a \le t \le b$, then

1. $\frac{d\vec{C}}{dt} = \vec{0}$, where $\vec{c}$ is a constant vector.
2. $\frac{d\left(C\vec{F}(t)\right)}{dt} = C\ \frac{d\vec{F}}{dt}$, where C is a constant.
3. $\frac{d(\vec{F}(t)\pm\vec{G}(t))}{dt} = \frac{d\vec{F}}{dt} \pm \frac{d\vec{G}}{dt}$
4. $\frac{d\left(f(t)\vec{F}(t)\right)}{dt} = f(t)\ \frac{d\vec{F}}{dt} + \frac{df}{dt}\vec{F}(t)$
5. $\frac{d(\vec{F}(t)\cdot\vec{G}(t))}{dt} = \frac{d\vec{F}}{dt}\cdot\vec{G}(t) + \vec{F}(t)\cdot\frac{d\vec{G}}{dt}$
6. $\frac{d(\vec{F}(t)\times\vec{G}(t))}{dt} = \frac{d\vec{F}}{dt}\times\vec{G}(t) + \vec{F}(t)\times\frac{d\vec{G}}{dt}$
7. $\frac{d}{dt}[\vec{F}(t),\ \vec{G}(t),\ \vec{H}(t)] = \left[\frac{d\vec{F}}{dt},\ \vec{G}(t),\ \vec{H}(t)\right] + \left[\vec{F}(t),\ \frac{d\vec{G}}{dt},\ \vec{H}(t)\right] + \left[\vec{F}(t),\ \vec{G}(t),\ \frac{d\vec{H}}{dt}\right]$
8. $\frac{d}{dt}\left[\vec{F}(t)\times\left(\vec{G}(t)\times\vec{H}(t)\right)\right] = \left[\frac{d\vec{F}}{dt}\times\ (\vec{G}(t)\times\ \vec{H}(t))\right] + \left[\vec{F}(t)\times\left(\frac{d\vec{G}}{dt}\times\ \vec{H}(t)\right)\right] + \left[\vec{F}(t)\times\left(\vec{G}(t)\times\frac{d\vec{H}}{dt}\right)\right]$
9. If $\vec{F}(t)$ is differentiable function of t and $t = t(s)$ is differentiable function then $\frac{d\vec{F}}{ds} = \frac{d\vec{F}}{dt}\frac{dt}{ds}$.

Observations:

(i) If $\vec{F}(t)$ has a constant magnitude, then $\vec{F}\cdot\frac{d\vec{F}}{dt} = 0$. For $\vec{F}(t)\cdot\vec{F}(t) = [\vec{F}(t)]^2 = constant$,
implying $\vec{F}\cdot\frac{d\vec{F}}{dt} = 0$ or $\vec{F}\perp\frac{d\vec{F}}{dt}$.

(ii) If $\vec{F}(t)$ has a constant (fixed) direction, then $\vec{F}\times\frac{d\vec{F}}{dt} = \vec{0}$.
Let $\vec{F}(t) = f(t)\vec{G}(t)$, where $\vec{G}(t)$ is a unit vector in the direction of $\vec{F}(t)$.
$\therefore\ \frac{d\vec{F}}{dt} = \frac{d\left(f(t)\vec{G}(t)\right)}{dt} = f(t)\ \frac{d\vec{G}}{dt} + \frac{df}{dt}\vec{G}(t) = \frac{df}{dt}\vec{G}(t)$ $\quad\left(since,\ \vec{G}\ is\ a\ constant, so\ \frac{d\vec{G}}{dt} = \vec{0}\right)$
and $\vec{F}\times\frac{d\vec{F}}{dt} = f(t)\vec{G}(t)\times\frac{df}{dt}\vec{G}(t) = f(t)\frac{df}{dt}\left(\vec{G}(t)\times\vec{G}(t)\right) = \vec{0}$ $\quad(since,\ \vec{G}\times\vec{G} = \vec{0})$

Theorem 1: Derivative of a constant vector is a zero vector. A vector is said to be constant if both its magnitude and direction are constant (fixed).

Proof: Let $\vec{r} = \vec{c}$ be a constant vector, then $\vec{r} + \delta\vec{r} = \vec{c}$.

On subtraction, $\delta\vec{r} = \vec{0}$.Which further implies that $\frac{\delta\vec{r}}{\delta t} = \vec{0}$.

Implying, $\underset{\delta t \to 0}{Lt} \frac{\delta \vec{r}}{\delta t} = \vec{0}$ i.e. $\frac{d\vec{r}}{dt} = \vec{0}$.

Theorem 2: The necessary and sufficient condition for the vector function $\vec{F}$ of a single variable t to have constant magnitude is $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$.

Proof:

Necessary condition: Suppose $\vec{F}$ has constant magnitude, so $\vec{F}(t) \cdot \vec{F}(t) = [\vec{F}(t)]^2 = constant$.

=> $\frac{d}{dt}(\vec{F} \cdot \vec{F}) = 0$ i.e. $\vec{F} \cdot \frac{d\vec{F}}{dt} + \frac{d\vec{F}}{dt} \cdot \vec{F} = 0$

=> $2\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$ i.e. $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$.

Sufficient condition: Suppose $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$ => $2\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$

=> $\vec{F} \cdot \frac{d\vec{F}}{dt} + \frac{d\vec{F}}{dt} \cdot \vec{F} = 0$ => $\frac{d}{dt}(\vec{F} \cdot \vec{F}) = 0$

=> $\vec{F} \cdot \vec{F} = constant$ => $|\vec{F}|^2 = constant$

Therefore $\vec{F}$ has a constant magnitude.

Theorem 3: The necessary and sufficient condition for the vector function $\vec{F}$ of a single variable t to have a constant direction is $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$.

Proof: Suppose that $\vec{f}$ is a unit vector in the direction of $\vec{F}$ and $F = |\vec{F}|$, then $\vec{f} = \frac{\vec{F}}{F}$ i.e.

$$\vec{F} = F\vec{f} \quad \ldots (1)$$

And $$\frac{d\vec{F}}{dt} = F\frac{d\vec{f}}{dt} + \frac{dF}{dt}\vec{f} \quad \ldots (2)$$

Thus $\vec{F} \times \frac{d\vec{F}}{dt} = F\vec{f} \times (F\frac{d\vec{f}}{dt} + \frac{dF}{dt}\vec{f})$ (using (1) and (2))

$= F^2\vec{f} \times \frac{d\vec{f}}{dt} + F\frac{dF}{dt}(\vec{f} \times \vec{f})$

$= F^2\vec{f} \times \frac{d\vec{f}}{dt}$ (since $\vec{f} \times \vec{f} = \vec{0}$) ... (3)

Necessary condition: Suppose $\vec{F}$ has a constant direction, then $\vec{f}$ has a constant direction and constant magnitude. So $\frac{d\vec{f}}{dt} = \vec{0}$. Thus from (3), $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$.

Sufficient condition : Suppose that $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$.

Then by (3), $F^2\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$ i.e. $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$ ... (4)

Since $\vec{f}$ has a constant magnitude, so, by theorem 2, $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$ ... (5)

Form (4) and (5), $\frac{d\vec{f}}{dt} = 0$.

Which implies $\vec{f}$ is a constant vector i.e. $\vec{f}$ has a constant direction. Hence $\vec{F}$ has a constant direction.

Example 1: Show that if $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$ where $\vec{a}, \vec{b}$ and ω are constants, then

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \text{ and } \vec{r} \times \frac{d\vec{r}}{dt} = -\omega(\vec{a} \times \vec{b}).$$

Solution: Given $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t$

Differentiating w. r. to t, $\frac{d\vec{r}}{dt} = \vec{a}\omega\cos\omega t - \vec{b}\omega\sin\omega t$

Again differentiating w. r. to t, $\frac{d^2\vec{r}}{dt^2} = -\vec{a}\,\omega^2\sin\omega t - \vec{b}\,\omega^2\cos\omega t$

$$= -\,\omega^2(\vec{a}\sin\omega t + \vec{b}\cos\omega t) = -\omega^2\vec{r}$$

Also $\vec{r} \times \frac{d\vec{r}}{dt} = (\vec{a}\sin\omega t + \vec{b}\cos\omega t) \times (\vec{a}\omega\cos\omega t - \vec{b}\omega\sin\omega t)$

$$= (\vec{a}\times\vec{a})\omega\sin\omega t\cos\omega t + (\vec{b}\times\vec{a})\omega\cos^2\omega t - (\vec{a}\times\vec{b})\omega\sin^2\omega t - (\vec{b}\times\vec{b})\omega\,\sin\omega t\cos\omega t$$

$$= -(\vec{a}\times\vec{b})\omega(\cos^2\omega t + \sin^2\omega t) \qquad (\textit{since } \vec{a}\times\vec{a} = \vec{b}\times\vec{b} = \vec{0})$$

$$= -(\vec{a}\times\vec{b})\omega = -\omega(\vec{a}\times\vec{b}).$$

Example 2: If $\vec{a} = x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$ and $\vec{b} = 2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k}$, find $\frac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b})$ at (1, 0, -2).

Solution: Here $\vec{a}\times\vec{b} = (x^2yz\,\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}) \times (2z\,\hat{i} + y\,\hat{j} - x^2\,\hat{k})$

$$= x^2y^2z\,\hat{k} + x^4yz\,\hat{j} + 4xz^4\,\hat{k} + 2x^3z^3\,\hat{i} + 2xz^3\,\hat{j} - xyz^2\,\hat{i}$$

$$(\because\ \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0} \textit{ and } \hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{i} = -\hat{k} \textit{ etc.})$$

$$= (2x^3z^3 - xyz^2)\hat{i} + (x^4yz + 2xz^3)\hat{j} + (x^2y^2z + 4xz^4)\hat{k}$$

Now $\frac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b}) = \frac{\partial^2}{\partial x\partial y}\left[(2x^3z^3 - xyz^2)\hat{i} + (x^4yz + 2xz^3)\hat{j} + (x^2y^2z + 4xz^4)\hat{k}\right]$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left[(2x^3z^3 - xyz^2)\hat{i} + (x^4yz + 2xz^3)\hat{j} + (x^2y^2z + 4xz^4)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\left[-xz^2\,\hat{i} + x^4z\,\hat{j} + 2x^2yz\,\hat{k}\right] = \left[-z^2\,\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}\right]$$

At the point (1, 0, -2) $\frac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b}) = -4\hat{i} - 8\hat{j}$

Example 3: If $\vec{P} = 5t^2\hat{i} + t^3\hat{j} - t\hat{k}$ and $\vec{Q} = 2\sin t\,\hat{i} - \cos t\,\hat{j} + 5t\hat{k}$, then find (*a*) $\frac{d}{dt}(\vec{P}\cdot\vec{Q})$ (*b*) $\frac{d}{dt}(\vec{P}\times\vec{Q})$.

Solution: Consider $\vec{P} = 5t^2\hat{i} + t^3\hat{j} - t\hat{k}$ and $\vec{Q} = 2\sin t\,\hat{i} - \cos t\,\hat{j} + 5t\hat{k}$

So $\frac{d\vec{P}}{dt} = 10t\,\hat{i} + 3t^2\hat{j} - \hat{k}$ and $\frac{d\vec{Q}}{dt} = 2\cos t\,\hat{i} + \sin t\,\hat{j} + 5\hat{k}$

a) $\frac{d}{dt}(\vec{P}\cdot\vec{Q}) = \frac{d\vec{P}}{dt}\cdot\vec{Q} + \vec{P}\cdot\frac{d\vec{Q}}{dt}$

$$= (10t\,\hat{i} + 3t^2\hat{j} - \hat{k})\cdot(2\sin t\,\hat{i} - \cos t\,\hat{j} + 5t\hat{k})$$

$$+(5t^2\hat{i} + t^3\hat{j} - t\hat{k})\cdot(2\cos t\,\hat{i} + \sin t\,\hat{j} + 5\hat{k})$$

$$= 20\,t\sin t - 3t^2\cos t - 5t + 10\,t^2\cos t + t^3\sin t - 5t$$

$$= t^3\sin t + 7t^2\cos t + 20\,t\sin t - 10\,t$$