# EXACT DIFFERENTIAL EQUATIONS

A project report submitted in partial fulfillment for the award of **Bachelor of Science (B.Sc.)**

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Submitted by

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# CERTIFICATE

This is to certified that the project entitled **"Exact Differential Equation"** submitted by B. SATISH for the award of Bachelor of Science (B.Sc.), Andhra University, Visakhapatnam, during the year 2019-20 is genuine record of the work done by him my supervision.

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Date :

# DICLARATION

I, hereby declare that project entitled **"Exact Differential Equation"** is an original work done by me and submitted to the Department of mathematics and Statistics, Mrs.AVN College, Visakhapatnam, for the fulfillment of the 6th Semester end examination. I also declare, that this or part of it has not been submitted to any other college for the award of degree .

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# 1.Chapter 1

**Introduction:** The next type of first order differential equations that we'll be looking at is exact differential equations. Before we get into the full details behind solving exact differential equations it's probably best to work an example that will help to show us just what an exact differential equation is. It will also show some of the behind the scenes details that we usually don't bother with in the solution process.

The vast majority of the following example will not be done in any of the remaining examples and the work that we will put into the remaining examples will not be shown in this example. The whole point behind this example is to show you just what an exact differential equation is, how we use this fact to arrive at a solution and why the process works as it does. The majority of the actual solution details will be shown in a later example.

*Example 1* :Solve the following differential equation.

$$2xy-9x^2+(2y+x^2+1)\frac{dy}{dx}=0$$

**Solution:**

Let's start off by supposing that somewhere out there in the world is a function $\Psi(x,y)$ that we can find. For this example the function that we need is

$$\Psi(x,y)=y^2+(x^2+1)y-3x^3$$

Do not worry at this point about where this function came from and how we found it. Finding the function, $\Psi(x,y)$, that is needed for any particular differential equation is where the vast majority of the work for these problems lies. As stated earlier however, the point of this example is to show you why the solution process works rather than showing you the actual solution process. We will see how to find this function in the next example, so at this point do not worry about how to find it, simply accept that it can be found and that we've done that for this particular differential equation.

Now, take some partial derivatives of the function.

$$\Psi_x=2xy-9x^2$$

$$\Psi_y=2y+x^2+1$$

Now, compare these partial derivatives to the differential equation and you'll notice that with these we can now write the differential equation as.

$$\Psi'_x+\Psi'_y\frac{dy}{dx}=0 \qquad (1)$$

Now, recall from your multi-variable calculus class (probably Calculus III), that (1) is nothing more than the following derivative (you'll need the multi-variable chain rule for this…).

$$\frac{d}{dx}(\Psi(x,y(x)))$$

So, the differential equation can now be written as

$$\frac{d}{dx}((\Psi(x,y(x)))=0$$

Now, if the ordinary (not partial…) derivative of something is zero, that something must have been a constant to start with. In other words, we've got to have $\Psi(x,y)=c$ Or,

$$y^2+(x^2+1)y-3x^3=c$$

This then is an implicit solution for our differential equation! If we had an initial condition we could solve for cc. We could also find an explicit solution if we wanted to, but we'll hold off on that until the next example.

Okay, so what did we learn from the last example? Let's look at things a little more generally. Suppose that we have the following differential equation.

$$M(x,y)+N(x,y)\frac{dy}{dx}==0 \qquad (2)$$

Note that it's important that it must be in this form! There must be an "= 0" on one side and the sign separating the two terms must be a "+". Now, if there is a function somewhere out there in the world, $\Psi(x,y)$, so that,

$$\Psi_x = M(x,y) \qquad \text{and} \qquad \Psi_y = N(x,y)$$

then we call the differential equation **exact**. In these cases we can write the differential equation as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0 \tag{3}$$

Then using the chain rule from your Multivariable Calculus class we can further reduce the differential equation to the following derivative,

$$\frac{d}{dx}(\Psi(x,y(x))) = 0$$

The (implicit) solution to an exact differential equation is then

$$\Psi(x,y) = c \tag{4}$$

Well, it's the solution provided we can find $\Psi(x,y)$ anyway.

Therefore, once we have the function we can always just jump straight to (4) to get an implicit solution to our differential equation.

Finding the function $\Psi(x,y)$ is clearly the central task in determining if a differential equation is exact and in finding its solution. As we will see, finding $\Psi(x,y)$ can be a somewhat lengthy process in which there is the chance of mistakes. Therefore, it would be nice if there was some simple test that we could use before even starting to see if a differential equation is exact or not. This will be especially useful if it turns out that the differential equation is not exact, since in this case $\Psi(x,y)$ will not exist. It would be a waste of time to try and find a nonexistent function!

So, let's see if we can find a test for exact differential equations. Let's start with (2) and assume that the differential equation is in fact exact. Since its exact we know that somewhere out there is a function $\Psi(x,y)$ that satisfies

$$\Psi_x=M$$

$$\Psi_y=N$$

Now, provided $\Psi(x,y)$ is continuous and its first order derivatives are also continuous we know that

$$\Psi_{xy}=\Psi_{yx}$$

However, we also have the following.

$$\Psi_{xy}=(\Psi_x)_y=(M)_y=M_y$$

$$\Psi_{yx}=(\Psi_y)_x=(N)_x=N_x$$

Therefore, if a differential equation is exact and $\Psi(x,y)$ meets all of its continuity conditions we must have.

$$M_y=N_x \qquad (5)$$

Likewise, if (5) is not true there is no way for the differential equation to be exact.

Therefore, we will use (5) as a test for exact differential equations. If (5) is true we will assume that the differential equation is exact and that $\Psi(x,y)$ meets all of its continuity conditions and proceed with finding it. Note that for all the examples here the continuity conditions will be met and so this won't be an issue.

Okay, let's go back and rework the first example. This time we will use the example to show how to find $\Psi(x,y)$. We'll also add in an initial condition to the problem.